# Dissipative Design of PI-Observers for Nonlinear Systems

Jaime A. Moreno*

Eléctrica y Computación, Instituto de Ingeniería
Universidad Nacional Autónoma de México, 04510 México D.F., Mexico
JMorenoP@ii.unam.mx

**Abstract.** A new dissipative method to design observers for a large class of nonlinear systems has been introduced recently by the author. It generalizes and includes several well-known observer design methods in the literature. In this paper a procedure to design Proportional-Integral (PI) Observers based on the dissipativity theory is presented. Properties of these observers are discussed.

## 1 Introduction

Recently [1, 2] the author has proposed a Dissipative Design of Observers for nonlinear systems that can be transformed into the form

$$\Sigma : \{\dot{x} = Ax + G\psi(\sigma) + \varphi(t, y, u) \ , \ y = Cx \ , \ \sigma = Hx \quad x(0) = x_0 \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state $x$, $u \in \mathbb{R}^m$ is a known input, $y \in \mathbb{R}^p$ is the measured output, and $\sigma \in \mathbb{R}^r$ is a (not necessarily measured) linear function of the state. $\varphi(t, y, u)$ is an arbitrary nonlinear function of the time, the input and the output. $\psi(\sigma)$ is a $q$-dimensional vector that depends on the variable $\sigma$. $\psi$ and $\varphi$ are assumed to be locally Lipschitz in $\sigma$ or $y$, continuous in $u$, and piecewise continuous in $t$, so that existence and uniqueness of solutions is guaranteed. It will be assumed that the trajectories of interest of $\Sigma$ are defined for all the time, i.e. there are no finite escape times.

The design can in the most important cases be reduced to Linear Matrix Inequalities (LMI), that are numerically very well behaved, and have become standard in the field. This method generalizes and encompases several other design methods, as the High-Gain methodology [3, 4], and the observers for Lipschitz nonlinear systems [5], well-known in the literature.

The objective of this paper is to present two further results for dissipative observers. First a general method to robustify such observers is presented in the context of so called proportional (P) observers. This can be useful to improve the robustness to certain perturbations. As an example, it will be shown that the classical first order sliding mode observers [6, 7] and the robust observers

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[8] can be included in the dissipative methodology. The same basic idea for designing proportional observers can be extended to designing Proportional-Integral (PI) Observers. PI-Observers are very well known in the literature, in particular for linear systems [9–13], because of their robustness properties against constant perturbations. This is basically a consequence of the internal model principle, since the integral action represents an internal model of unknown constant perturbations, a fact that is widely used in robust regulation. Although most of the literature on PI-Observers is related to linear systems, there are some proposals to extend the idea to nonlinear systems. They use some specific nonlinear observer design methods, in particular, the design for Lipschitz systems [14, 15], and propose linear PI gains.

Our design idea consists basically of two steps. In the first one a nonlinear observer is designed by the dissipative method for the unperturbed system, so that many standard observer designs are included here. In a second step a robustifying nonlinear P or PI additional term is added to the basic nonlinear observer, providing improved performance properties. These terms are so constructed, that they inject more dissipation to the dissipative observer. For this kind of observers the separation property derived in [16] for the basic dissipative observers can be extended to systems with constant perturbations.

## 2 Preliminaries

### 2.1 Dissipative systems

In this work the stability properties of dissipative systems will be used for the design of observers for systems that can be represented as the feedback interconnection of a dynamical linear time invariant (LTI) system in the forward loop and a memoryless nonlinearity in the feedback loop. From the general dissipativity theory [17] (see also [1, 2]) the following results are of relevance here.

Consider the LTI continuous time system $\dot{x} = Ax + Bu$, $y = Cx$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, and $y \in \mathbb{R}^m$ are the state, the input and the output vectors, respectively. Let us consider quadratic $\omega(y,u) = y^T Q y + 2y^T S u + u^T R u$, where $Q \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times q}$, $R \in \mathbb{R}^{q\times q}$, and $Q$, $R$ symmetric. System $\Sigma_L$ is said to be ( ) with respect to the supply rate $\omega(y,u)$, or for short $(Q,S,R)$-SSD, if there exist a matrix $P = P^T > 0$, and $\epsilon > 0$ such that

$$\begin{bmatrix} PA + A^T P + \epsilon P \,, & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} C^T Q C & C^T S \\ S^T C & R \end{bmatrix} \leq 0\,. \tag{2}$$

For quadratic systems, i.e. $m = q$, passivity corresponds to the supply rate $\omega(y,u) = y^T u$. Note that this definition assures the existence of a quadratic positive definite $V(x) = x^T P x$, such that along any trajectory of the system $\dot{V}(x(t)) \leq \omega(y(t), u(t)) - \epsilon V(x(t))$.

A time-varying memoryless nonlinearity $\psi : [0,\infty) \times \mathbb{R}^q \to \mathbb{R}^m$, $y = \psi(t,u)$, piecewise continuous in $t$ and locally Lipschitz in $u$, such that $\psi(t,0) = 0$, is said to be dissipative with respect to the supply rate $\omega(y,u)$, or for short $(Q,S,R)$-D, if it satisfies $\omega(\psi(t,u),u) \geq 0$, for every $t \geq 0$, and $u \in \mathbf{R}^q$. The classical sector conditions [18] for square nonlinearities, i.e. $m = q$, can be represented in this form. If $\psi$ is in the sector $[K_1, K_2]$, i.e. $(y - K_1 u)^T (K_2 u - y) \geq 0$, then it is $(Q,S,R)$-D, with $(Q,S,R) = \left(-I, \frac{1}{2}(K_1 + K_2), -\frac{1}{2}\left(K_1^T K_2 + K_2^T K_1\right)\right)$.

A generalization of the circle criterion of absolute stability for non square systems can be easily obtained, and it will be used in the sequel.

**Lemma 1.** *[1, 2] Consider the feedback interconnection*

$$\dot{x} = Ax + Bu \ , \ y = Cx \ , \ u = -\psi(t,y) \ , \quad x(0) = x_0 \tag{3}$$

*If the system $(C,A,B)$ is $\left(-R_N, S_N^T, -Q_N\right)$-SSD, then $x = 0$ for (3) is globally exponentially stable for every $(Q_N, S_N, R_N)$-D nonlinearity.*

### 2.2 A strong Lyapunov function

To analyze the convergence properties of PI-Observers it will be required to study conditions for the asymptotic stability of the interconnection of a nonlinear globally asymptotically stable system in the forward loop with an integrator in the feedback. This general class of systems is very important in adaptive control and identification [19] and it is usually studied, from a passive perspective, as the negative feedback interconnection of two passive subsystems. In this case the sum of the storage functions constitutes a weak Lyapunov function, that is, one whose time derivative is only negative semidefinite, even in the cases when asymptotic stability can be assured. We will be interested here in a special class with a time-invariant interconnection. The novelty of our result here is that we will provide explicit conditions for the global exponential stability of the whole system and we will give a *strong* Lyapunov function that ensures this.

Consider the following system

$$\Xi : \{\dot{x} = f(x,t) + Bk(z) \ , \ \dot{z} = Cx \ , \ x \in \mathbf{R}^n \ , z \in \mathbf{R}^p \tag{4}$$

where $f(x,t)$ is locally Lipschitz in $x$ and measurable in $t$, $k : \mathbf{R}^p \to \mathbf{R}^p$ is locally Lipschitz continuous and it is the gradient of a scalar, positive definite, decrescent, radially unbounded, continuously differentiable function $W(z)$, i.e. $k^T(z) = \partial_z W(z)$, $k(0) = 0$, and $B$, $C$ are constant matrices of appropriate dimensions. Assume that $f(0,t) = 0$, and that the system $\dot{x} = f(x,t)$ has zero as a globally uniformly asymptotically stable equilibrium point, and that there is a quadratic Lyapunov function $V(x) = x^T Px$, with $P$ symmetric and positive definite, such that $\dot{V}(x) = \partial_x V(x) f(x,t) \leq -\epsilon V(x)$, with $\epsilon > 0$. From a passivity approach it follows that, if $B = -P^{-1}C^T$ then the function $V^*(x,z) = V(x) + W(z)$ satisfies

$$\dot{V}^*(x,z) \leq -\epsilon V(x) - x^T C^T k(z) + \partial_z W(z) Cx = -\epsilon V(x) \ .$$

From this property it follows the uniform stability of the equilibrium point, the boundedness of the trajectories and the asymptotic convergence to zero of $x$. To assure the uniform asymptotic stability of the origin further conditions. e.g. the Theorem of Matrosov, have to be used. $V^*$ is therefore a *weak* Lyapunov function for the system. Here we propose a *strong* Lyapunov function, that assures the GUAS of the origin, under some additional assumptions.

**Proposition 1.** *Consider the system (4) satisfying the given conditions. Suppose further that $f(x,t)$ is globally Lipschitz in $x$ uniformly in $t$, that the Jacobian matrix of $k(z)$ is continuous, and it is uniformly upper and lower bounded, and that $C$ has full row rank. Under these conditions*

$$U(x,z) = \delta V^*(x,z) + x^T C^T k(z) \ ,$$

*with $\delta > 0$ a sufficiently large constant, is a strong Lyapunov function for the system, and the equilibrium point is globally exponentially stable.*

*Proof.* We show first that for $\delta$ sufficiently high $U$ is positive definite and decrescent, i.e. there exist $K_\infty$ functions $\alpha_1$, $\alpha_2$ such that $\alpha_1(\|(x,z)\|) \leq U(x,z) \leq \alpha_2(\|(x,z)\|)$. Note that

$$U(x,z) = \delta x^T Px + x^T C^T k(z) + \delta W(z) = \delta x^T Px + x^T C^T H(tz)\, z + \frac{1}{2}\delta z^T H(tz)\, z \, ,$$

since from the mean value theorem it follows that $W(z) = \frac{1}{2} z^T H(tz)\, z$, $k(z) = H(\tau z)\, z$, for some $t, \tau \in (0,1)$, and where $H(z) = \frac{\partial^2 W(z)}{\partial z^2}$ is the Hessian matrix of $W(z)$. Since, by assumption, $c_1 \mathbf{I} \leq H(z) \leq c_2 \mathbf{I}$ for all $z \in \mathbf{R}^p$ and some positive constants $c_1, c_2$, it follows easily that $U$ is positive definite and decrescent for some value of $\delta$ sufficiently large. Next we show that the derivative of $U$ is negative definite. Recall that by the Lipschitz hypothesis $\|f(x,t)\| \leq \lambda \|x\|$, for all $t \geq 0$ and $x \in \mathbf{R}^n$.

$$\dot{U}(x,z) \leq -\begin{bmatrix} x & z \end{bmatrix} \Theta \begin{bmatrix} x \\ z \end{bmatrix} \leq -\alpha_3(\|(x,z)\|) \ .$$

where

$$\Theta = \begin{bmatrix} \delta \epsilon P - C^T H(z)\, C & \frac{1}{2}\lambda C^T H(\tau z) \\ \frac{1}{2}\lambda H(\tau z)\, C & H(\tau z)\, C P^{-1} C^T H(\tau z) \end{bmatrix} \ .$$

It follows that $\dot{U}$ is negative definite for some $\delta$ sufficiently large. Moreover, since $\alpha_1$, $\alpha_2$ and $\alpha_3$ are all quadratic functions, global exponential stability follows.

## 3 Dissipative Observer Design

The Dissipative Design of Observers is a method recently proposed in [1,2]. Its basic idea is to decompose the observer error dynamics into dissipative subsystems, and using the dissipative theory, design the output injection in such a way, that the error dynamics converges. For the particular class of systems described by the form (1) the results are particularly suitable for calculations. A full order observer for $\Sigma$ of the form

$$\Omega : \begin{cases} \dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) + G\psi(\hat{\sigma} + N(\hat{y} - y)) + \varphi(t, y, u) \ , & \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x} \ , \ \hat{\sigma} = H\hat{x} \end{cases} \quad (5)$$

is proposed, where matrices $L \in \mathbb{R}^{n \times p}$, and $N \in \mathbb{R}^{r \times p}$ have to be designed. Defining the state estimation error by $e \triangleq \hat{x} - x$, the output estimation error by $\tilde{y} \triangleq \hat{y} - y$, and the functional estimation error by $\tilde{\sigma} \triangleq \hat{\sigma} - \sigma$, the dynamics of $e$ can be written as

$$\Xi : \{\dot{e} = A_L e + G\nu \ , \ z = H_N e \ , \ \nu = -\phi(z, \sigma) \ , \quad (6)$$

where $A_L \triangleq A + LC$, $H_N \triangleq H + NC$, $z \triangleq H_N e$, and a new nonlinearity $\phi(z, \sigma) \triangleq \psi(\sigma) - \psi(\sigma + z)$. Note that $\phi(0, \sigma) = 0$ for all $\sigma$. Note that, in general, the error dynamics (6) is not autonomous, since it is driven by the system (1) through the linear function of the state $\sigma = Hx$. $\phi$ is therefore a time varying nonlinearity, whose time variation depends on the state trajectory of the plant.

The observer design consists in finding matrices $L$ and $N$, if they exist, so that $\Xi$ satisfies the conditions of Lemma 1. For this it is necessary to assume that the nonlinearity $\phi$ satisfy one or several supply rates $\omega$: $\phi$ is $(Q_i, S_i, R_i)$-dissipative for some finite set of non positive semidefinite quadratic forms $\omega_i(\phi, z) = \phi^T Q_i \phi + 2\phi^T S_i z + z^T R_i z \geq 0$, for all $\sigma$, for $i = 1, 2, \cdots, M$.

Note that if $\phi$ satisfies the assumption, then it is $\sum_{i=1}^{M} \theta_i (Q_i, S_i, R_i)$-dissipative for every $\theta_i \geq 0$, i.e. $\phi$ is dissipative with respect to the supply rate $\omega(\phi, z) = \sum_{i=1}^{M} \theta_i \omega_i(\phi, z)$. In this case the design is as follows:

**Theorem 1.** $\phi$ $L$ $N$ $\theta = (\theta_1, \cdots, \theta_M)$ $\theta_i \geq 0$ $\Xi$ $(-R_\theta, S_\theta^T, -Q_\theta)$ $(Q_\theta, S_\theta, R_\theta) = \sum_{i=1}^{M} \theta_i (Q_i, S_i, R_i)$ $P = P^T > 0$ $K$ $W$ $\epsilon > 0$

$$\begin{bmatrix} PA_L + A_L^T P + \epsilon P + H_N^T R_\theta H_N \ , & PG - H_N^T S_\theta^T \\ G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \leq 0 \ , \quad (7)$$

$\Omega$ $\Sigma$ $\kappa, \gamma > 0$ $\|e(t)\| \leq \kappa \|e(0)\| \exp(-\gamma t)$ $e(0)$ $t \geq 0$ $V(e) = e^T P e$ $\Xi$ $\dot{V} \leq -\epsilon V(e)$

The observer design relies on finding (if they exist) matrices $L$ and $N$, a vector $\theta = (\theta_1, \cdots, \theta_M)$, $\theta_i \geq 0$, a matrix $P = P^T > 0$, and $\epsilon > 0$ such that the inequality (7) is satisfied. In general this is a nonlinear matrix inequality feasibility problem. However, if $N$ is fixed, then it is a Linear Matrix Inequality (LMI) feasibility problem, for which a solution can be effectively found by several algorithms in the literature [20].

The proposed method generalizes, unifies and improves several methods previously proposed in the literature. Some of them are [1]: (i) The Circle criterion design: It is easy to see that this method generalizes and improves the one proposed originally in [21], and further developed in [22] (and references therein): our design is valid for non-square systems, the nonlinearities are of general type, and can be described by several sector conditions. (ii) Lipschitz observer design: proposed recently by [5]. (iii) High-Gain observer design: The well-known high-gain observer (HG) design [3] is a special case of the one proposed here.

## 4 A Robustified Dissipative Observer Design

If the conditions of Theorem 1 are satisfied then (5) is a convergent observer for the plant. However it is useful to be able to add more dissipation to the observer error dynamics. In this section, based on the original design, a redesigned, more dissipative observer will be proposed.

Consider the error dynamics of the observer (6). The design pursued was obtained assigning the right dissipativity to the LTI map $\nu \rightarrow z$ to compensate the nonlinearity $\phi$. Similar to the idea of adding damping control, well-known in the literature [23], an additional dissipation term will be added to the error dynamics. For this consider an additional "input" $v$ to the error system $\dot{e} = A_L e - G\phi(H_N e, \sigma) + v$, $\tilde{y} = Ce$. Since $V(e) = e^T Pe$ is a Lyapunov function for the system we have for $\Xi_v$ that $\dot{V} \leq -\epsilon V(e) + e^T Pv$. Selecting $v = -P^{-1}C^T Q^T \chi(Q\tilde{y}, t)$, where $Q$ is an arbitrary matrix, and $\chi$ satisfies $\xi^T \chi(\xi, t) \geq 0$, $\forall \xi$ , $\forall t \geq 0$, then

$$\dot{V} \leq -\epsilon V(e) - e^T PP^{-1}C^T Q^T \chi(Q\tilde{y}, t) = -\epsilon V(e) - \tilde{y}^T Q^T \chi(Q\tilde{y}, t) \leq -\epsilon V(e) \ .$$

This shows that an extra dissipative term has been added to the error equation. This enhances the convergence properties of the observation error, what can be used for robustification of the observer. Note that with this selection the map $r \rightarrow w$ of the error equation

$$\Xi_v : \begin{cases} \dot{e} = A_L e - G\phi(z, \sigma) + P^{-1}C^T Q^T r \ , \\ \tilde{y} = Ce \ , \ z = H_N e \ , \ w = Q\tilde{y} \ , \ r = -\chi(w, t) \ , \end{cases} \tag{8}$$

becomes strictly passive. To realize this extra term in the error equation, the observer has to be redesigned as

$$\Omega_R : \begin{cases} \dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) + G\psi(\hat{\sigma} + N(\hat{y} - y)) + \varphi(t, y, u) - P^{-1}C^T Q^T \chi(Q\tilde{y}, t) \ , \\ \hat{y} = C\hat{x} \ , \ \hat{\sigma} = H\hat{x} \end{cases} \tag{9}$$

This proves the following

**Lemma 2.** $\chi(\xi,t)$ $\xi^T\chi(\xi,t) \geq 0$ , $\forall\xi$ , $\forall t \geq 0$ $\Omega_R$ ( ) $\Sigma$ ( ) ( )

There are many possible selections for the function $\chi$. For example, selecting $Q = I$ and $\chi(\tilde{y}) = \mathrm{sign}(\tilde{y})$ one recovers the sliding mode observer [6, 7] and the robust observers [8], with global convergence properties.

The previous lemma shows that the state estimation error of the (robustified) dissipative observer (9) converges exponentially, with a convergence rate that is independent of the trajectories of the plant. The following Lemma states a robustness property of the observation error against external perturbations and the convergence of the nonlinear signals.

**Lemma 3.** $\phi(z,\sigma)$ $\chi(w,t)$ $z$ $w$ $\phi(z(t),\sigma(t))$ $\chi(w(t),t)$ $L_1$ $\phi(z,\sigma)$ $\chi(w,t)$ $\delta_1(t)$ $\delta_2(t)$ $\delta_1(t)$ $\delta_2(t)$ ( )

## 5 A Dissipative Proportional-Integral Observer Design

The observer designed in Theorem 1 is , since only a static nonlinear function of the estimation error is injected. It is well known that the injection of an integral term of the estimation error greatly improves the robustness properties of the observer. In what follows, a (nonlinear) proportional-integral term will be included in the dissipative observer, and the properties of such observer will be studied using the dissipativity theory.

A ( ) for system (1) is a dynamical system $\Omega_{PI}$ that has as inputs the input $u$ and the output $y$ of $\Sigma$, and its output $\hat{x}$ is an estimation of the state $x$ of $\Sigma$. A full order PI observer for $\Sigma$ of the form

$$\Omega_{PI}: \begin{cases} \dot{\hat{x}} = A\hat{x} + L\tilde{y} + G\psi(\hat{\sigma} + N\tilde{y}) + \varphi(t,y,u) + E\left[\varkappa_I(\xi) + \varkappa_P(K\tilde{y})\right] , \\ \dot{\xi} = K\tilde{y} \ , \ \hat{y} = C\hat{x} \ , \ \hat{\sigma} = H\hat{x} \ , \ \tilde{y} = \hat{y} - y \ , \ \xi(0) = \xi_0 \end{cases} \tag{10}$$

is proposed, where matrices $L \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{r\times p}$, $K \in \mathbb{R}^{q\times p}$, and $E \in \mathbb{R}^{n\times q}$, and the functions $\varkappa_I : \mathbb{R}^q \to \mathbb{R}^q$, and $\varkappa_P : \mathbb{R}^q \to \mathbb{R}^q$ have to be designed. The dynamics of the error system $\Xi_{PI}$ can be written as the feedback interconnection of two systems:

$$\Xi_{PI}: \begin{cases} \Xi_1 : \left\{\dot{e} = A_L e - G\phi(H_N e,\sigma) + Ev \ , \ \tilde{\gamma} = KCe \ , \right. \\ \Xi_2 : \left\{\dot{\xi} = \tilde{\gamma} \ , \ v = \varkappa_I(\xi) + \varkappa_P(\tilde{\gamma}) \ , \right. \end{cases} \tag{11}$$

The objective of the design is to render the (closed) set $\{(e, \xi) \in \mathbb{R}^{n \times q} \mid e = 0\}$ asymptotically stable for system $\Xi_{PI}$. Since $V_1(e) = e^T P e$ is a Lyapunov function for the system we have for $\Xi_1$ that $\dot{V}_1 \leq -\epsilon V_1(e) - e^T P E(-v)$. If $K$ and $E$ are selected such that $e^T P E = -\tilde{\gamma}^T = -e^T C^T K^T$, then it follows that $\Xi_1$ is strictly state passive from $(-v) \to \tilde{\gamma}$.

It will be shown in the sequel that if $\varkappa_I(\cdot)$ and $\varkappa_P(\cdot)$ are selected appropriately, then the subsystem $\Xi_2$ is also passive from $\tilde{\gamma} \to v$. Consider a $C^1$ function $V_2(\xi) \geq 0$ for all $\xi \in \mathbb{R}^q$, and $V_2(0) = 0$. Select $\varkappa_I(\xi) = (\partial_\xi V_2(\xi))^T$. Then it is clear that $V_2(\xi) = \int_0^\xi \varkappa_I(z) \cdot dz$. Moreover, if $\varkappa_P(\cdot)$ is such that $\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} \geq 0$ for all $\tilde{\gamma}$, then along the trajectories of $\Xi_2$ it is satisfied

$$\dot{V}_2(\xi) = \varkappa_I^T(\xi)\dot{\xi} = \varkappa_I^T(\xi)\tilde{\gamma} = -\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} + v^T\tilde{\gamma} \leq v^T\tilde{\gamma} .$$

It follows then that the time derivative of the storage function $V(e, \xi) = V_1(e) + V_2(\xi)$ along the solutions of $\Xi_{PI}$ is $\dot{V}(e, \xi) \leq -\epsilon V_1(e) - \tilde{\gamma}^T v - \varkappa_P^T(\tilde{\gamma})\tilde{\gamma} + v^T\tilde{\gamma} \leq -\epsilon V_1(e) - \varkappa_P^T(\tilde{\gamma})\tilde{\gamma}$. This ensures that $e(t) \to 0$ as $t \to \infty$. Moreover, if $V_2(\xi)$ is radially unbounded, then the state $(e, \xi)$ will be bounded.

If convergence of the equilibrium point $(e, \xi) = 0$ is desired, further conditions are required. A set of such conditions are given in the next theorem, together with a strong Lyapunov function to ensure this.

**Theorem 2.**
$\psi(\sigma)$ $\varkappa_P(\tilde{\gamma})$ $K \in \mathbb{R}^{q \times p}$ $KC$
$C^2$ $\mathbb{R}^q$ $V_2(\xi)$
$V_2(0) = 0$ $E = -P^{-1}C^T K^T$ $\varkappa_I(\xi) = \left(\frac{\partial V_2(\xi)}{\partial \xi}\right)^T$ $\varkappa_P^T(\tilde{\gamma})\tilde{\gamma} \geq 0$
$\Omega_{PI}$ ( ) $\Sigma$
$U(e, \xi) = \delta e^T P e + e^T C^T K^T \varkappa_I(\xi) + \delta V_2(\xi)$
$\delta$

Let us rewrite the system as $\dot{e} = f(e, \sigma) - P^{-1}C^T K^T \varkappa_I(\xi)$, $\dot{\xi} = KCe$, where $f(e, \sigma) \triangleq A_L e - G\phi(H_N e, \sigma) - P^{-1}C^T K^T \varkappa_P(KCe)$. From the Hypothesis it follows that $f(e, \sigma)$ is globally Lipschitz uniformly in $\sigma$. Consider $V(e) = e^T P e$. Its time derivative along the solutions of system $\dot{e} = f(e, \sigma)$ is $\dot{V}(e) = e^T P f(e, \sigma) \leq -\epsilon V(e)$. The result then follows directly from Prop. 1.

A particular, but important, case is the one when $\varkappa_I(\xi)$ and $\varkappa_P(\tilde{\gamma})$ are linear functions, i.e. $\varkappa_I(\xi) = K_I \xi$ , $K_I = K_I^T > 0$, $\varkappa_P(\tilde{\gamma}) = K_P \tilde{\gamma}$ , $K_P > 0$. In this case $V_2(\xi) = \frac{1}{2}\xi^T K_I \xi$. However, it is also possible to use nonlinear and discontinuous functions $\varkappa_P(\tilde{\gamma})$, as in the previous section.

## 6 Robustness properties of the PI-Observer

Consider the plant (1) with a constant input perturbation

$$\Sigma : \{\dot{x} = Ax + G\psi(\sigma) + \varphi(t, y, u) + Dw \ , \ y = Cx \ , \ \sigma = Hx \tag{12}$$

where $w \in \mathbb{R}^q$ is a constant (or slowly time-varying) perturbation. If a PI-Observer (10) is designed, then the error dynamics becomes

$$\Xi_{PI} : \begin{cases} \Xi_1 : \{\dot{e} = A_L e - G\phi(H_N e, \sigma) + Ev - Dw \ , \ \tilde{\gamma} = KCe \ , \\ \Xi_2 : \left\{\dot{\xi} = \tilde{\gamma} \ , \ v = \varkappa_I(\xi) + \varkappa_P(\tilde{\gamma}) \right. \ , \end{cases}$$

or also, taking $E = -P^{-1}C^T K^T = D$, $\dot{e} = f(e,\sigma) + D(\varkappa_I(\xi) - w)$, $\dot{\xi} = KCe$ , where $f(e,\sigma) \triangleq A_L e - G\phi(H_N e, \sigma) - P^{-1}C^T K^T \varkappa_P(KCe)$. Defining $\bar{\xi} = \varkappa_I^{-1}(w)$, that exists for every $w$, since $\varkappa_I$ is globally invertible, and introducing $e_\xi = \xi - \bar{\xi}$, the error dynamics becomes

$$\Xi_{PI} : \left\{\dot{e} = f(e,\sigma) + D\left(\varkappa_I\left(\bar{\xi} + e_\xi\right) - \varkappa_I\left(\bar{\xi}\right)\right) \ , \ \dot{e}_\xi = KCe + \zeta \ , \right.$$

where $\zeta = d\bar{\xi}/dt = \left(\partial_\xi \varkappa_I\left(\bar{\xi}\right)\right)^{-1} \dot{w}$ is related to the time derivative of $w$. Then from Lemma 3 and Theorem 2 it follows easily the following convergence result for the PI-Observer.

**Theorem 3.** ( ) $D =$
$-P^{-1}C^T K^T$ $E = D$
$\Omega_{PI}$ ( ) $\Sigma$ $\hat{x} \to x$
$\xi \to \bar{\xi}$ $w$ ( )
$w$ $(e, e_\xi)$
$w$

When the (unknown) perturbation is constant, its value can be determined from the state of the integral term, since $w = \varkappa_I\left(\bar{\xi}\right)$. If $w$ changes slowly, this estimation is correct up to a small error. When the matching condition $D = -P^{-1}C^T K^T$ is satisfied, the effect of the perturbation can be compensated exactly by the integral term of the PI-observer.

## 7 Conclusions

A new dissipative method to design observers for a large class of nonlinear systems has been introduced recently by the author. It generalizes and includes several well-known observer design methods in the literature. In this paper a procedure to increase the robustness of these dissipative observers is presented. The increased performance can be obtained by means of (nonlinear) proportional or proportional-integral terms. These PI-Observers have the usual properties of systems with integral terms, that are robust against constant perturbations, and they can be used, in principle, for robust regulation purposes.

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